Thesis defense

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Main results

- Spectrum of quantum KdV hierarchy in the semi-classical limit
  with Dymarsky, Sugishita, Pavlenko
  [arXiv:2208.01062]
- Information geometry and holographic correlators
  with Sivaramakrishnan, Bohra
  [JHEP 04 (2022) 037]
- Classical codes and chiral CFTs at higher genus
  with Henriksson, McPeak
  [JHEP 05 (2022) 159]
- Narain CFTs and Quantum Codes at Higher Genus
  with Henriksson, McPeak
  [arXiv:2205.00025]

Other work (not included in thesis)

- A quantum annealing based algorithm to calculate distance of a Quantum Error Detection Code
  with Dymarsky, Ismail
  [in prep]
- Characterizing Error Mitigation by Symmetry Verification in Quantum Approximate Optimization Algorithm (QAOA)
  with Larson, Galda, Shaydulin
  [2204.05852]
- Understanding the role of boundary conditions in Modular Hamiltonian of conformal scalar field
  with Dymarsky, Shapere
  [internal note]

When does an isolated quantum system thermalize?

- Eigenstate Thermalization Hypothesis (ETH) gives us a criterion
- Look at matrix elements of a probe observable O in energy eigenstates

$$\langle E_i|O|E_j\rangle = \delta_{ij} f_O(E_i) + e^{-S(E_i+E_j)/2} g(E_i, E_j) r_{ij}$$

- Expectation values of O are given by the canonical ensemble at late times

$$\langle \psi(t)|O|\psi(t)\rangle = Tr\ Oe^{-\beta H}$$

[Srednicki '94, Deutsch '91]

What happens when the system has many conserved charges Q_{2k-1}

- Generalized Eigenstate Thermalization Hypothesis (GETH)
- Look at matrix elements of a probe observable O in mutual eigenstates $|E_j\rangle$ of all the charges

$$\langle E_i|O|E_j\rangle = \delta_{ij} f_O(E_i) + e^{-S(E_i+E_j)/2} g(E_i, E_j) r_{ij}$$

- Expectation values of O are given by the Generalized Gibbs Ensemble (GGE) at late times

$$\langle \psi(t)|O|\psi(t)\rangle = Tr\ O\ e^{-\sum_k \mu_{2k-1} Q_{2k-1}}$$

[Rigol, Dunjko, Olshanii '08]

qKdV Hierarchy in 2d CFTs

- In any integrable 2d CFT, you can construct an infinite set of mutually commuting conserved charges
- classical kdV hierarchy

$$Q_1^{cl} = \int d\phi u(\phi), \quad Q_3^{cl} = \int d\phi u(\phi)^2,$$

- Quantum kdV hierarchy

$$Q_1 = \int d\phi T, \quad Q_3 = \int d\phi : T^2 :,$$

qKdV Hierarchy in 2d CFTs

- These charges give us flows in phase space

$$\dot{u} = \{Q_1^{cl}, u\}, \quad \dot{u} = \{Q_3^{cl}, u\} = 6u\partial u - \partial^3 u$$

- Quantum version

$$\dot{T} = [Q_1, T],$$
$$\dot{T} = [Q_3, T] = -3\partial(TT) - \frac{c-1}{6}\partial^3 T$$

- In a seminal work, the existence and relation to integrability was shown

$$[Q_{2k-1}, Q_{2l-1}] = 0$$

[Bazhanov, Lukyanov, Zamalodchikov '96]

Eigenvalue problem for qKdV charges

- The states $L_{-m_1}...L_{-m_k}|\Delta\rangle$ form a basis of the Verma module
- There is a particular basis in the Verma module which is eigenbasis of qKdV charges

$$|\psi\rangle = L_{-m_1}...L_{m_k}|\Delta\rangle + ...$$
$$Q_{2n-1}|\psi\rangle = \lambda_{2n-1}|\psi\rangle$$

- n_k is defined in the free boson representation of the CFT: n_k counts the number of times k appears in the set $\{m_i\}$

$$|\{n_k\},\Delta\rangle = |\{m_i\},\Delta\rangle$$

- Example

$$L_{-2}^2 L_{-1} \quad \text{is} \quad n_2 = 2, \quad n_1 = 1$$

Main Result: Spectrum of qkdV charges

- The calculation of the eigenvalues λ_{2n-1} for all charges Q_{2n-1} in a perturbative $1/c$ expansion.

$$Q_{2n-1} = \Delta^n + c^{n-1} \sum_k n_k f_1(k, \Delta)$$
$$+ c^{n-2} \left(\sum_k n_k^2 g_2(k, \Delta) + \sum_{k,p} n_k n_p g_1(k, p, \Delta) + \sum_k g_0(k, \Delta) \right)$$
$$+ O(c^{n-3})$$

- Obtained closed form expressions for f_1, g_2, g_1, g_0 for all n.

Broad strategy

- We will first calculate the classical KdV charges Q^{cl}_{2n-1}
- Large c expansion in the quantum theory $\sim$ expansion in action variables I_k in the classical theory.

$$Q^{cl}_{2n-1} = h^n + \sum_k f_1(k) I_k + \sum_k f_2(k) I_k^2 + \ldots$$

- **Semi-classical quantization rule:**:
Multiply Q^{cl}_{2n-1} by $\left(\frac{c}{24}\right)^n$ and

$$I_k \longrightarrow \frac{24}{c}\left(n_k + \frac{1}{2}\right), \quad h \longrightarrow \frac{24}{c}\left(\Delta + \frac{c}{24}\right)$$

- **Constraint from Modular covariance**

$$\langle Q_{2n-1} \rangle_\beta = \text{modular covariant with weight 2n}$$

Semi-classical quantization and large c

- Holographically relevant $1/c$ expansion
- Intuition for semi-classical quantization: action variables for hydrogen atom quantized
- Additional constraints to completely fix quantum result up to 2^{nd} order in expansion: modular covariance/action of Q on primaries

[Maloney, Ng, Ross and Tsiares '19]
[Dorey, Dunning, Negro, Tateo'19]

Novikov's method

[Novikov '74]

- To study solutions $u(x)$ of

$$\frac{c}{24}\{Q_{2k-1}, u\} = 0$$

  Study the spectral problem of

$$-\frac{d^2}{dx^2}\psi + u\psi = \lambda\psi$$

- **Inverse scattering problem**: Given spectrum of the Schrodinger equation
- Try and reconstruct the potential $u(x)$
- This problem was solved by Novikov for periodic $u(x)$.

Turn Novikov's analysis Perturbative

[Novikov '74]

- Perform the appropriate phase space integrals perturbatively

$$I_k = \frac{i}{\pi} \oint_{a_i} dp \log \lambda$$

- The conserved quantities

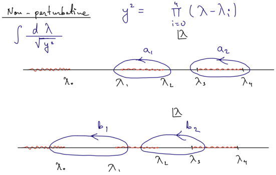
$$Q_{2n-1} = -\frac{\Gamma(n+1)(\Gamma(1/2))}{(\Gamma(n+1/2))(2\pi i)} \oint_{\infty} dp \lambda^{n-1/2}$$

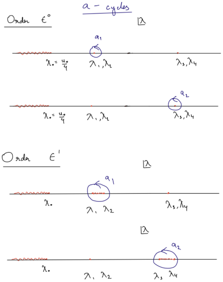
- Our approach: Do it perturbatively in distance between λ_i
- Reduces higher genus phase space integrals to torus ones which are tractable.
- It allows us to get the expansion

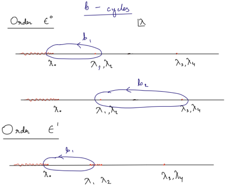
$$Q^{cl}_{2n-1} = h^n + \sum_k f_1(k) I_k + \sum_k f_2(k) I_k^2 + \ldots$$

How to do this for higher genus surfaces using pictures

Perturbative parameter: distance between roots of hyper-elliptic curve







Conclusion

- Large c spectrum of qkdV from semi-classical quantization
- Developed methods to calculate classical spectrum
  $Q^{cl}_{2n-1} = h^n + \sum_k f_1(k) I_k + \sum_k f_2(k) I_k^2 + ...$
  to all orders
- Raises questions:
- What are the modular properties of Z_{GGE}?
- Can you use this spectrum to find universal hydrodynamic properties of integrable $2d$ CFTs?
- kdV charged black holes

Modular invariance of CFT partition functions

- $(t_E, x) \sim (t_E + \beta, x + 2\pi)$ is the same as $z \sim z + 1 \sim z + \tau$

$$Z = \sum_{h,\bar{h}\ \in\ \text{states}} q^{h-\frac{c}{24}} \bar{q}^{\bar{h}-\frac{\bar{c}}{24}}, \text{ where } \quad q = e^{2i\pi\tau}, \quad \bar{q} = e^{-2i\pi\bar{\tau}}.$$

- The partition can be sliced in different ways
- Modular invariance:

$$Z\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) = Z(\tau, \bar{\tau}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Modular bootstrap

- Non-perturbative method to determine space of allowed theories from symmetry and unitarity
- Demand $Z(\tau,\bar{\tau})$ is invariant under:

$$T:\tau \longrightarrow \tau+1, \quad S:\tau \longrightarrow \frac{-1}{\tau}$$

$$Z(\tau,\bar{\tau}) = \chi_{vac}(\tau,\bar{\tau}) + \sum_{h,\bar{h}} d\left(h,\bar{h}\right)\chi_{h,\bar{h}}(\tau,\bar{\tau})$$

- Hellerman bound for pure 3d gravity : What is the largest gap Δ_1 compatible with modular invariance (assuming Virasoro characters) ? $\Delta_1 \sim c/6 + 0.474$

Motivation: taking bootstrap programme to higher genus

- Constraints easy to solve
- Using the correspondence between codes and CFTs, examples were constructed of "fake" theories which are modular invariant, can be expanded in (Virasoro) characters with non-negative integral coefficients and with unique vacuum
- Many examples of non-isomorphic theories sharing the same partition function. Modular bootstrap cannot tell them apart.
- Non-chiral version of Milnor's famous example: Can you hear the shape of a drum?

[Dymarsky, Shapere '21]

Classical code

- What is an $[n, k, d]$ classical linear code?
  Collection of 2^k code-words
  Each codeword $c \in F^n(2)$
  A bit-flip on $\lfloor (d-1)/2 \rfloor$ bits can be corrected
- Weight $w(c) \sim$ no. of 1's
- An example: Hamming [8,4,4] code :
  1011 is encoded into 01100110
  0000 is encoded into 00000000
  Upto 4 bits can be corrupted

Code CFTs: a testing ground for Modular Bootstrap approach to solve CFTs

- Associated with a classical code is an enumerator polynomial:

$$W_{\mathcal{C}}(x_0, x_1) = \sum_{c \in \mathcal{C}} x_0^{n-w(c)} x_1^{w(c)} .$$

- Construction A by Leech and Sloane relates a Euclidean Lattice $\Lambda(\mathcal{C})$ to a code $\mathcal{C}$

$$\Theta_{\Lambda(\mathcal{C})}(\tau) = W_{\mathcal{C}}(\theta_3(q^2), \theta_2(q^2)),$$

- This allows one to define a 2-d CFT with central charge c living on this lattice, with torus partition function

$$Z(\tau) = \frac{\Theta_\Lambda(\tau)}{\eta(\tau)^c} .$$

Classical linear self-dual code –> Euclidean self-dual lattice

- Associated with a classical code is an enumerator polynomial:

$$W_{\mathcal{C}}(x_0, x_1) = \sum_{c \in \mathcal{C}} x_0^{n-w(c)} x_1^{w(c)}$$

- Associated with a Euclidean lattice is lattice theta series:

$$\Theta_\Lambda(\tau) = \sum_{v \in \Lambda} q^{v^2/2}, \qquad q = e^{2\pi i \tau}$$

- Construction A by Leech and Sloane relates a Euclidean Lattice $\Lambda(\mathcal{C})$ to a code $\mathcal{C}$

$$\Theta_{\Lambda(\mathcal{C})}(\tau) = W_{\mathcal{C}}(\theta_3(q^2), \theta_2(q^2))$$

- "code CFT" partition function

$$Z(\tau) = \frac{\Theta_\Lambda(\tau)}{\eta(\tau)^c}$$

A testing ground for Bootstrap approach to solve CFTs

- Modular transformations are written very simply in code variables:

$$S: x_0 \mapsto \frac{x_0 + x_1}{\sqrt{2}}, \qquad x_1 \mapsto \frac{x_0 - x_1}{\sqrt{2}}$$

$$T: x_1 \mapsto i x_1$$

- These can be easily solved for and solutions to this for $c = 24$ give 190 possible code CFTs
- But there are only 9 known Lattice CFTs you get by Construction A
- How do you rule out the rest via symmetries?

Higher genus modular invariance

- Define Period matrix :

$$\oint_{a_i} \omega_j = \delta_{ij}, \quad \oint_{b_i} \omega_j = \Omega_{ij}.$$

- $Z(\Omega_{ij})$ is invariant under

$$\Omega \mapsto \tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$$

Genus 2 lattice theta series

- Genus 2 lattice theta series is well defined:

$$\Theta_{\Lambda}^{g=2}(\Omega) = \sum_{\vec{v}_1, \vec{v}_2 \in \Lambda} q^{\frac{\vec{v}_1 \cdot \vec{v}_1}{2}} r^{\vec{v}_1 \cdot \vec{v}_2} s^{\frac{\vec{v}_2 \cdot \vec{v}_2}{2}},$$

  with the the modular parameters q, r, s are defined as

$$q = e^{2\pi i \Omega_{11}}, \qquad r = e^{2\pi i \Omega_{12}}, \qquad s = e^{2\pi i \Omega_{22}}.$$

- So is the bi-weight enumerator polynomial:

$$W_{\mathcal{C}}^{(2)}(x_0, x_1, x_2, x_3) = \sum_{c_1, c_2 \in \mathcal{C}} x_0^{n + c_1 \cdot c_2 - w(c_1) - w(c_2)} x_1^{w(c_2) - c_1 \cdot c_2} x_2^{w(c_1) - c_1 \cdot c_2} x_3^{c_1 \cdot c_2}.$$

- This screams at you: Apply Construction A here

Higher genus transformations in code variables

- The theta map: theta constants of second order characteristic
  $x_i \longrightarrow \theta \begin{bmatrix} \vec{c}_i/2 \\ \vec{0} \end{bmatrix} (0, 2\Omega)$
- Genus 2 modular transformations:

$$\begin{aligned}
T_{g=2}: \quad & x_0 \mapsto x_0, && x_1 \mapsto x_1, && x_2 \mapsto i x_2, && x_3 \mapsto i x_3, \\
R_{g=2}: \quad & x_0 \mapsto x_0, && x_1 \mapsto x_3, && x_2 \mapsto x_2, && x_3 \mapsto x_1, \\
D_{g=2}: \quad & x_0 \mapsto \frac{x_0 + x_2}{\sqrt{2}}, && x_1 \mapsto \frac{x_0 - x_2}{\sqrt{2}}, && x_2 \mapsto \frac{x_1 + x_3}{\sqrt{2}}, && x_3 \mapsto \frac{x_1 - x_3}{\sqrt{2}}
\end{aligned}$$

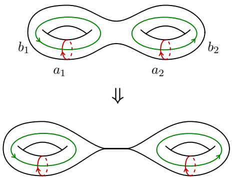
- Degeneration limit: identity exchange: polynomials factorize:

$$W_{\mathcal{C}}^{(g=2)}(x_i) \mapsto W_{\mathcal{C}}^{(1)}(x_i) W_{\mathcal{C}}^{(1)}(y_i)$$

where

$$x_0 \to x_0^2, \quad x_1 \to x_0 x_1, \quad x_2 \to x_0 x_1, \quad x_3 \to x_1^2$$

Algorithm



- Write all possible homogeneous polynomials consistent with symmetries
- Under Degeneration polynomials factorize into consistent genus 1 partition functions:

$$W_{\mathcal{C}}^{(g=2)}(x_i) \mapsto W_{\mathcal{C}}^{(1)}(x_i) W_{\mathcal{C}}^{(1)}(y_i),$$

where

$$x_0 \to x_0^2, \quad x_1 \to x_0 x_1, \quad x_2 \to x_0 x_1, \quad x_3 \to x_1^2 .$$

- Demand positive degeneracy of codewords

Chiral results

- Example: Chiral $c = 24$:
  There are 190 genus 1 polynomials.
  29 come from consistent genus 2 polynomials
  21 at genus 3.
  9 Codes and 24 self dual lattices
- We also reproduce the above results, and provide an interpretation in terms of degeneration of Siegel modular forms upto genus 3.

$$\Theta_{\Lambda}^{g=2} = E_4^3 + a_1\psi_{12} + a_2\chi_{12}.$$

[Runge '94]
[Gaberdiel, Volpato '08]

Non-Chiral results

- genus 2 modular transformations act linearly on 10 code variables
- Example: Code CFTs with $n = 4$:
  At genus 1: 20 polynomials.
  At genus 2: 45 polynomials but only 10 factorize.
  9 of these polynomials derive from real codes, leaving only one fake
- Determined the full polynomial ring that generates invariant polynomials
- Non-chiral resolution of Milnors example: all $n = 7$ and $n = 8$ iso-spectral theories have different genus 2 partition functions

Great people I worked with!

"All that is gold does not glitter,
Not all those who wander are lost;
The old that is strong does not wither,
Deep roots are not reached by the frost." - J.R.R. Tolkien, The Fellowship of the Ring

- Anatoly Dymarsky
- Sotaro Sugishita, Nagoya University
- Brian McPeak and Johan Henrikson, University of Pisa
- Allic Sivaramkrishnan and Hardik Bohra, University of Kentucky

- String theory group: Sumit Das, Al Shapere
- Condensed matter theory group: Ganpathy Murthy, Ribhu Kaul

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